

Synchronization of Anticonformist Populations

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Abstract

Hipsters purport to transcend mainstream style, yet they all dress alike. Counterintuitively, anticonformist individuals that strive to avoid looking like the majority end up synchronizing with each other and looking the same. In a recent preprint on the arXiv, mathematician Jonathan Touboul tries to explain this perplexing behavior using a 2-state model. The model describes a large population of two types of individuals, conformist and anticonformist, that will respectively tend to align their states with or against the average state of the majority. Additionally, the individuals perceive the state of the majority with some delay; that is, an individual makes her decision based on the majority trend some time ago. Touboul finds that, in majority anticonformist populations, the anticonformists remain unsynchronized for small values of this delay. But, as the delay is increased, a critical value is reached above which the anticonformist states align and oscillate periodically and synchronously.

In our paper, we will reproduce Touboul's results, using simulations and analytical calculations. Focusing on the special case of an entirely anticonformist population, we will run simulations of the time evolution of the population for varying values of the information delay to determine the critical delay threshold for synchronization. We intend to calculate the critical delay value analytically as well. Also, by defining an order parameter for the system, we will compare numerical and analytical calculations of the dependence of the asymptotic value of the order parameter on the time delay.

Introduction

Hipsters strive to defy normalcy but, often they appose the norm in the same way. This lack of originality is a salient feature that many systems in nature exhibit. For instance, the random misfiring of neurons in the brain become synchronized during epileptic seizures¹. In late October of 2014, Jonathan Touboul, a French mathematician who is studying the behavior of neural networks, posted a paper on the arXiv entitled "The hipster effect: When anticonformists all look the same."² In this paper, Touboul develops a toy model that captures the phenomenology of the spontaneous synchronization of anticonformist (*i.e.* hipster) elements. The model is related to phenomena observed in other systems where elements make decisions based on a majority trend, arising in areas such as psychology, spin glasses, and economics³.

Here, we reproduce the basic results of Touboul by using both simulation and an analytical approach. In a system composed entirely of hipsters, our simulations show the anticonformists undergo spontaneous synchronization above a critical threshold of the time delay with which they perceive the majority trend. Above this threshold, we observe the hipsters oscillate coherently. Using a Galerkin-like approach, we are able to analytically predict the frequency and amplitude response of the synchronized oscillation to the choice of the delay near critical value, in fantastic agreement with the simulations.

The Model

Touboul's model is a stochastic model that consists of a population of n independent, 2 state elements. Each element, i , is either in the state $S_i = -1$ or the state $S_i = 1$. The elements are one of two types: either conformist ($\epsilon_i = 1$) or anticonformist ($\epsilon_i = -1$). The conformist elements tend to orient themselves with the majority of the other elements while the anticonformists tend to orient opposite that of the majority. The probability of any individual at time t to change its state from S_i to $-S_i$ is given by a Poisson process with a rate:

$$\phi(t) = 1 + \tanh[-\epsilon_i \beta m(t - \tau) S_i(t)] \quad (1)$$

Where $\beta > 0$ is the inverse temperature (or inverse noise strength) and $m(t)$ is the majority trend

$$m(t) = \frac{1}{n} \sum_{i=1}^n S_i(t) . \quad (2)$$

¹M. Chavez, *et. al.*. *Spatio-temporal dynamics prior to neocortical seizures: amplitude versus phase couplings*. IEEE Transactions on Biomedical Engineering 2003; 50(5): 571–583.

²Jonathan Touboul. *The hipster effect: When anticonformists all look the same*. arXiv:1410.8001

³Damien Challet. *et. al.*. *Statistical mechanics of systems with heterogeneous agents: Minority Games*. arXiv:cond-mat/9904392

In order to account for the time it takes information to travel, each element perceives what the majority state at time t is at the delayed time $t + \tau$. That is, each element's tendency to switch is informed by the rest of the population at an earlier time. The parameter β effectively controls the noise level in the system. As β increases the sharpness of the transition rate function increases meaning the transition becomes less random. This rate function leads to a probability of switching states between time t and $t + \Delta t$ (for small Δt):

$$P(t, \Delta t) = \sinh(\phi(t) \Delta t) e^{-\phi(t) \Delta t} \quad (3)$$

Intuitively, if the population is predominantly comprised of conformists, the system will tend towards a steady state regardless of the time it takes each element to perceive the mean trend. Whatever the initial majority state is, each conformist element will tend to switch to that state, only further strengthening the trend. In the case of a majority anticonformist population however, each anticonformist element tends to orient itself against the mainstream trend which acts to weaken the integrity of the mainstream. If the anticonformist's perception of the mainstream trend is outdated, then the anticonformist could conceivably switch the state that is currently mainstream. This would act to strengthen the mainstream trend during the time it takes the elements perception to catch up (*i.e.* τ). Now, consider that every anticonformist element is doing this. Together they will all begin to switch away from the majority at the delayed time creating a new mainstream. Perhaps, there is a critical delay τ_c such that for $\tau > \tau_c$ the system will exhibit oscillatory mainstream trends whose frequency, f , is of order $1/\tau$.

In order to investigate this model further, we wrote our own MatLab simulations. Our simulations gave the same results that Touboul reports from his simulation in his paper. To focus our study, we considered a system that entirely consists of anticonformists. The simulations were carried out with $\beta = 2$, $n = 5 \times 10^3$, $\Delta t = 10^{-2}$, and the initial condition was chosen at random. Typical output results from our simulation are shown in figure 1. The simulation shows that there is in fact a τ_c such that for all $\tau \geq \tau_c$ the system exhibits

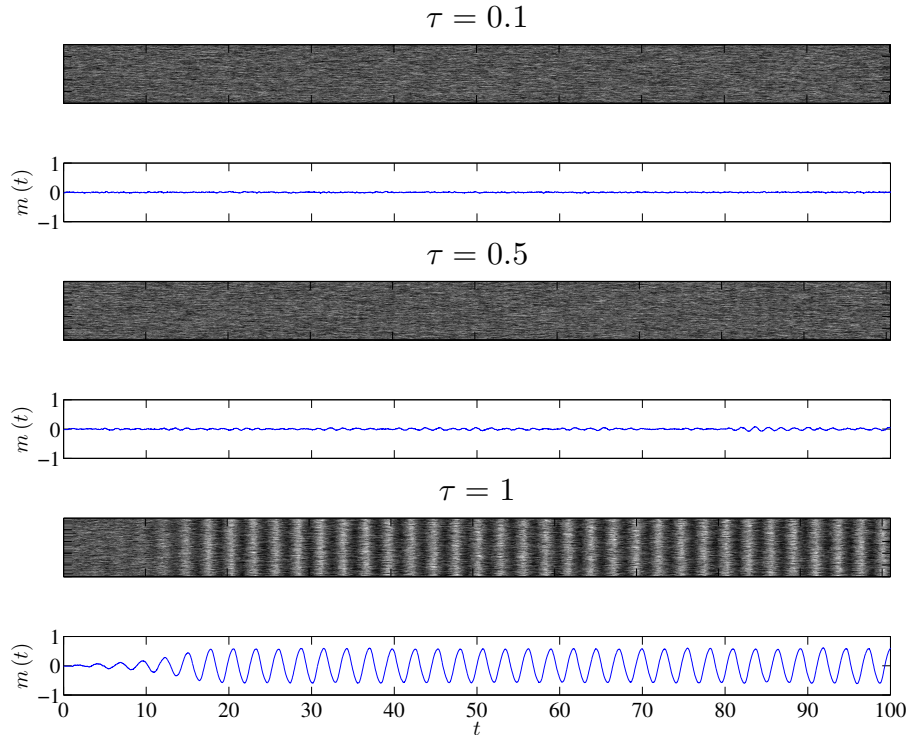


Figure 1: Sample simulation outputs for various values of τ . Values of τ are chosen to be well below τ_c , just below τ_c , and well above τ_c . Each pixel on the black and white images correspond to the state of an individual element. The plots below the images show the mean trend vs. time. Both the plots and the images are plotted on the same time scale.

periodic oscillations where as, below τ_c $m = 0$ is the stable solution. For τ just below τ_c there are patches of partially synchronized periodicity that emerge but, eventually decay. This weak bursting of synchronization can be seen in figure 2. Figure 2 is a spectrogram of the log of the power spectral density as a function of τ . The weak partial synchronization is evident by the broadband curve that is present before τ_c , represented by the red dotted line.

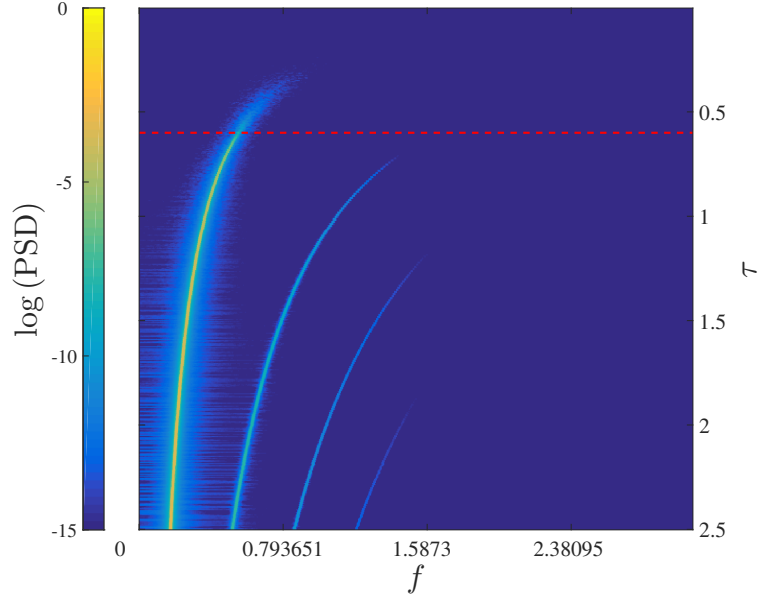


Figure 2: Log of the power spectral density of $m(t)$ for different values of τ . Dotted line is τ_c compute from linear stability analysis.

Analyzing the Model

According to Touboul, in the thermodynamic limit that the number of hipsters $n \rightarrow \infty$ and in the mean field approximation, the averaged trend $m(t)$ of the hipsters satisfies the differential equation

$$\dot{m}(t) = -2\left(m(t) + \tanh(\beta m(t - \tau))\right). \quad (4)$$

Note that because this is a delayed differential equation, the initial condition is a function on the time interval $[-\tau, 0]$. Thus, the initial condition belongs to an infinite-dimensional vector space and so do the solutions to this equation, unlike the finite-dimensional solutions of ordinary differential equations. Despite this complication, it is clear that $m(t) = 0$ for all t is a steady-state solution. Physically, this corresponds to the totally disordered state of the hipsters. We follow Touboul by investigating the linear stability of this solution.

We Taylor-expand the hyperbolic tangent about $m = 0$ to first order, yielding

$$\dot{m}(t) \approx -2\left(m(t) + \beta m(t - \tau)\right). \quad (5)$$

Next, we assume solutions of the form $m(t) = ae^{\lambda t}$. Substituting this into Eq. (5) and eliminating common factors of $ae^{\lambda t}$ gives the characteristic equation

$$\lambda = -2(1 + \beta e^{-\lambda\tau}) \quad (6)$$

for the eigenvalues of the system. Because of the exponential, there are an infinite number of complex solutions to this equation. What's more, if a single one of them has $\text{Re}[\lambda] > 0$, then 0 is an unstable equilibrium. We investigate this by substituting $\lambda = a + ib$ into Eq. (6) and taking real and imaginary parts. This gives the pair of equations

$$a = -2\left(1 + \beta e^{-a\tau} \cos(b\tau)\right), \quad (7)$$

$$b = 2\beta e^{-a\tau} \sin(b\tau). \quad (8)$$

Focusing on Eq. (7), we see that if τ is small enough, then $\cos(b\tau) \approx 1$ and because $\beta > 0$ and $e^{-a\tau} > 0$, we have $a < 0$. Thus, for small τ , the totally disordered solution $m(t) = 0$ is linearly stable.

As τ is increased, however, a Hopf bifurcation may occur, meaning a changes sign from negative to positive and the equilibrium solution is no longer stable. By definition, the Hopf bifurcation happens when the eigenvalues cross the imaginary axis, i.e. $a = 0$. Substituting this condition into Eqs. (7) and (8) and using some algebra and a trigonometric identity, we get the relation

$$\tau_c = \frac{-\tan^{-1}(\sqrt{\beta^2 - 1}) + k\pi}{2\sqrt{\beta^2 - 1}} \text{ for } k \text{ an integer.} \quad (9)$$

Thus, for a fixed β , Eq. (9) gives the critical delay value τ_c for which the $m = 0$ solution goes unstable. To compare with our simulations, we take $\beta = 2$, and $k = 1$ gives the first positive value of the critical time delay

$\tau_c \approx 0.605$. This value is plotted as the vertical dotted line in Figure 3, where the amplitude of the hipster oscillation versus τ computed from simulations is plotted as the solid curve in the upper graph. We see the analytical value of τ_c is exactly where the amplitude of the average trend oscillation significantly departs from 0. Evidently, above τ_c , the stable solution becomes a periodic oscillatory function whose average amplitude depends on τ .

We now take the calculation one order higher and compute the dependence of the frequency and amplitude of the oscillation on τ near the critical point. Empirically, we see the steady-state solution for the mean trend is periodic for $\tau > \tau_c$, so $m(t)$ may be expanded in a Fourier series. Also, from the power spectrum of $m(t)$ in Figure 2 obtained from simulations, we observe that near τ_c (dotted line), the first Fourier mode dominates the spectrum. Thus, for small $\tau - \tau_c$, we estimate $m(t)$ by only looking at the first term of its Fourier series,

$$m(t) \approx a \sin \omega t. \quad (10)$$

Now by requiring the solution to be self-consistent, i.e. forcing it to satisfy Eq. (4), we can obtain formulas for the dependence of a and ω on τ . Because we are interested in τ near τ_c we can also safely assume small oscillations for m . Then we can again expand the hyperbolic function in Eq. (4) about 0, this time to third order in m , yielding

$$\dot{m}(t) \approx -2 \left(m(t) + \beta m(t - \tau) - \frac{1}{3} \beta^3 m(t - \tau)^3 \right). \quad (11)$$

We then substitute Eq. (10) into Eq. (11), and in the spirit of the Galerkin method, we discard higher frequency Fourier terms that are generated by the cubic term, keeping only the first Fourier mode. After the application of some trigonometric identities, we respectively set the coefficients of the sine and cosine terms on both sides of the equation equal to each other, giving the system of equations

$$\omega a = \beta a \sin(\omega \tau) \left(2 - \frac{1}{2} \beta^2 a^2 \right) \quad (12)$$

$$2a = -\beta a \cos(\omega \tau) \left(2 - \frac{1}{2} \beta^2 a^2 \right) \quad (13)$$

which determine a and ω as functions of τ .

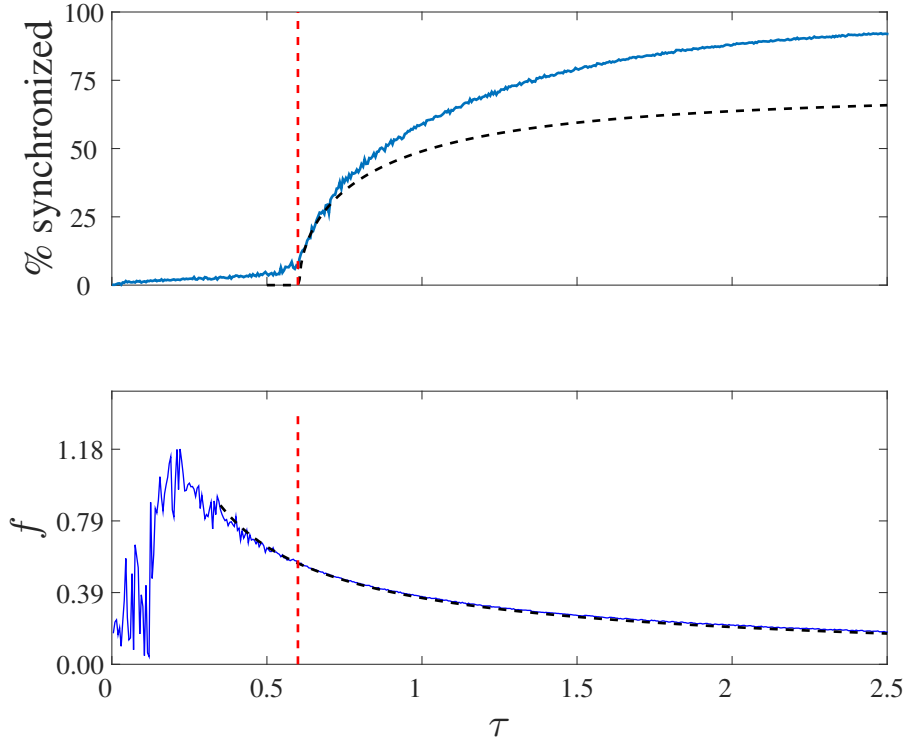


Figure 3: Amplitude (upper graph) and fundamental frequency (lower graph) of the mean trend oscillation as a function of the delay τ . The amplitude is expressed as a percentage of the population that is in sync. Simulation results are solid curves, analytical approximations are dotted curves. The vertical dotted line τ_c is derived from linear stability analysis.

Dividing Eqs. (12) and (13) gives an exact relationship between ω and τ ,

$$\tan(\omega \tau) = -\frac{\omega}{2}. \quad (14)$$

Remarkably, this relationship is completely unaffected by the noise β . For a given τ , this equation has infinitely many solutions for ω , so we choose the first positive solution as our value of $\omega(\tau)$, *i.e.*

$$\omega\tau = \pi - \tan^{-1}\left(\frac{\omega}{2}\right) \quad (15)$$

We compute $\omega(\tau)$ thus using a numerical root finder, and we compare this result with our simulations in the lower graph of Figure 3. We computed the fundamental frequency ω from our simulations by finding the lowest frequency peak of the Fourier transform of the simulated $m(t)$, and this is plotted versus τ as the solid line. Our analytical curve $\omega(\tau)$, the dotted line, agrees fantastically well with the simulations. Surprisingly, we also see agreement for $\tau < \tau_c$ and even values of τ well above τ_c . This may be explained by the fact that ω is independent of the noise, so regardless the β , the system may have oscillatory solution for all τ . However, these solutions are not stable below τ_c .

We can also implicitly compute the dependence of the oscillation amplitude a on τ . Looking at Eq. (13), we discard the trivial solution $a = 0$ and solve for a . In addition, by using Eq. (15) and using a trigonometric identity, we get

$$a = \frac{2}{\beta} \sqrt{1 - \frac{1}{\beta} \sqrt{1 + \frac{\omega^2}{4}}}. \quad (16)$$

While we cannot explicitly express a as $a(\tau)$, we know we can get ω from Eq. (14), so a truly only depends on τ . Eq. (16) can also be used to compute τ_c by setting $a = 0$. In fact, by substituting this condition into Eq. (16) and using Eq. (15), one arrives exactly at Eq. (9) for τ_c derived from linear stability analysis. Eq. (16) is plotted in the upper graph of Figure 3, showing that it agrees very well with the oscillation amplitude in the simulations near τ_c .

Conclusion

In conclusion, we were able to find spontaneous synchronization in Toubol's toy-model of hipsters. In a system consisting of many two-state agents that tend to align their own states with the opposite of the majority trend, we reproduced Touboul's surprising result that the agent-system experiences a phase transition from the expected completely disordered state to a synchronized, oscillatory state. This phase transition occurs as you increase the time delay with which the hipsters perceive the majority trend, τ , past a critical value τ_c . We were also able to reproduce Touboul's analytical calculation of τ_c , and were able to extend his results by finding the implicit dependence of the angular frequency ω and oscillation amplitude a on τ near τ_c . In particular, our function $\omega(\tau)$ seemed to be accurate far from τ_c as well, so we speculate that it is possible to show our relation for $\omega(\tau)$ is valid for all τ .

The synchronization experienced by the hipsters is very similar to other models exhibiting spontaneous synchronization, such as the Kuramoto model. Below a threshold of a tuning parameter, the system has a unique, stable, completely disordered solution that it reaches regardless of initial preparation. When the tuning parameter reaches a critical value, the system settles in an ordered solution that can be characterized by an order parameter. In our case, the amplitude of the hipster trend oscillation plays the role of the order parameter. This order parameter gradually increases from 0 as the relevant tuning parameter is increased past the critical point, and the shape of its turn on is a square-root, characteristic of the Kuramoto model and other models. We find it remarkable that these common traits of spontaneous synchronization can be found in Touboul's hipster model, in which every agent is acting precisely to avoid synchronization.