Dynamic Stability Of An Inverted Pendulum

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Abstract

For centuries the top of a pendulums path has been considered to be an unstable equilibrium point; however, in 1951, Pytor Kapitza experimentally showed that this point can be dynamically stable if the pendulums suspension is shaken periodically at a sufficiently high frequency. More general theoretical work has been done to generalize the form of the driving vibration even accounting for a form of stochastic, non-periodic, noise. Here, I will present a study of the strength of the stability of the periodically driven inverted pendulum as a function of amplitude and frequency as well as an investigation of a stochastically driven inverted pendulum.

1 Introduction

In physics, the pendulum has remained one of the standard textbook problems for centuries perhaps because it is both a familiar and a diverse system. If you only allow for small oscillations about the bottom, you recover simple harmonic motion but, in general, it is a non-linear system whose dynamics are complicated and include the presence of strange attractors [1]. For centuries the top of pendulums path was considered to be an unstable equilibrium point but, in 1951, Pytor Kapitza experimentally showed that this point can be dynamically stable if the pendulums suspension is shaken periodically at a sufficiently high frequency [3, 4].

Since Kapitza, many more studies (both theoretical and experimental) have been published with applications in the construction of sky-scrapers [2], robotics [7], as well as many other areas of technology. More general theoretical work has been done to generalize the form of the driving vibration even accounting for a form of stochastic, non-periodic, noise[6, 5]. The form of this noise is, however, not simple zero mean white noise.

Derivation of the equation of motion

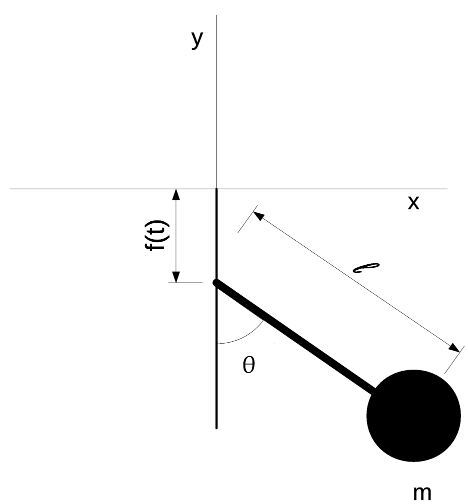


Figure 1: Diagram of a pendulum with all of the parameters labeled.

The inverted shaking pendulum (see figure 1) that is being investigated has a pivot position given by:

$$x = l \sin(\theta)$$

$$y = f(t) + l \cos(\theta)$$

where $f(t)$ in an arbitrary function of time. The Lagrangian, $\mathscr{L}$, is then,

$$\mathscr{L} = \frac{m}{2}\left[l^2\dot{\theta}^2 + \dot{f}^2 + L\dot{f}\dot{\theta}\sin(\theta)\right] + mgl\cos(\theta).$$

This Lagrangian together with the power function for linear angular drag

$$Q = \frac{\partial P}{\partial \dot{\theta}} = -bl^2\dot{\theta}$$

yields an equation of motion θ:

$$\Rightarrow \ddot{\theta} = -\left(\frac{g}{l} + \frac{\ddot{f}}{2l}\right)\sin(\theta) - \frac{b}{m}\dot{\theta}$$

Solving this differential equation to 2^{nd} order for $l = 1$ using trapezoidal method for the linear term, Adams-Bashforth for the non-linear, and explicit Euler for $\ddot{f}$ gives:

$$\Omega_{t+1} = \frac{1}{1+\frac{b\Delta t}{2}}\left[\left(1 - \frac{b\Delta t}{2}\right)\Omega_t - \Delta t\left(1 + \frac{1}{2}\ddot{f}\left(\frac{3}{2}\sin\theta_t - \frac{1}{2}\sin\theta_{t-1}\right)\right)\right]$$

$$\theta_{t+1} = \theta_t + \Delta t\,\Omega_{t+1}$$

2 MatLab simulation – periodic drive

Periodically shaking the pendulums pivot is known to stabilize $\theta = \pi$ for a sufficiently large frequency. To see this, let

$$f(t) = \epsilon\sin\omega t$$

Using MatLab to implement this periodic shake to the pivot shows that $\theta = \pi$ becomes stable for certain values of shake amplitude, ϵ, and frequency, ω. The MatLab code that was used to integrate the equation of motion can be seen in Appendix A of this document

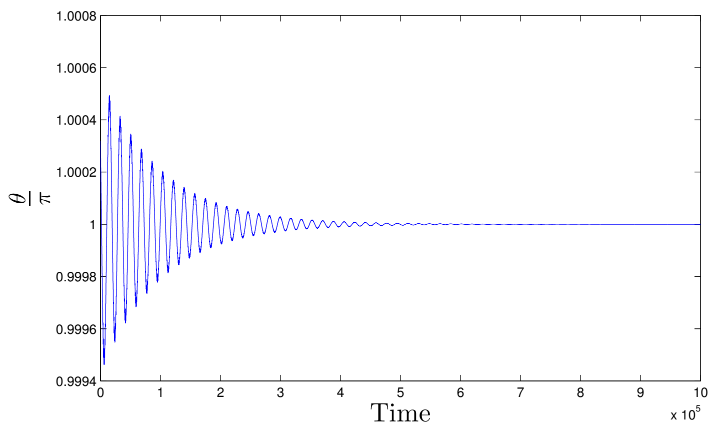


Figure 2: Sample MatLab time series of the pendulums angular position of a periodically driven pendulum for a sufficiently large of ω. Note that $\theta = \pi$ has become a stable solution.

Strength of stability

In order to investigate quantitatively the effects of ω, ϵ, and the noise strength, κ on the dynamically stable $\theta = \pi$ solution, a measure of stability that is not found by trial and error necessary. The region around $\theta = \pi$ where a pendulum – placed at rest – would remain in the inverted configuration (*i.e.* the region of initial stability) was used as this measure of stability. If the region of initial stability is large than, $\theta = \pi$ is a *very stable* equilibrium. Similarly, if the stable region is small than the equilibrium has relatively *weak stability*.

The calculation of the region of initial stability is done by simulating the time evolution of an ensemble of $N = 100$ initial conditions uniformly spread in the region $\theta = \pi \pm \alpha$. After a sufficiently long time the standard deviation, $\sigma_{t \gg t_o}$, of the ensemble is calculated. If the ensemble is contained within the region of initial stability than $\sigma_{t \gg t_o} < \sigma_{initial}$. However, if a portion of the ensemble is outside the stable region than, $\sigma_{t \gg t_o} > \sigma_{initial}$. The value of α where $\sigma_{t \gg t_o}$ begins growing is then the edge of the stable region. The width of the region of initial stability versus both ϵ and ω are shown in figure 3.

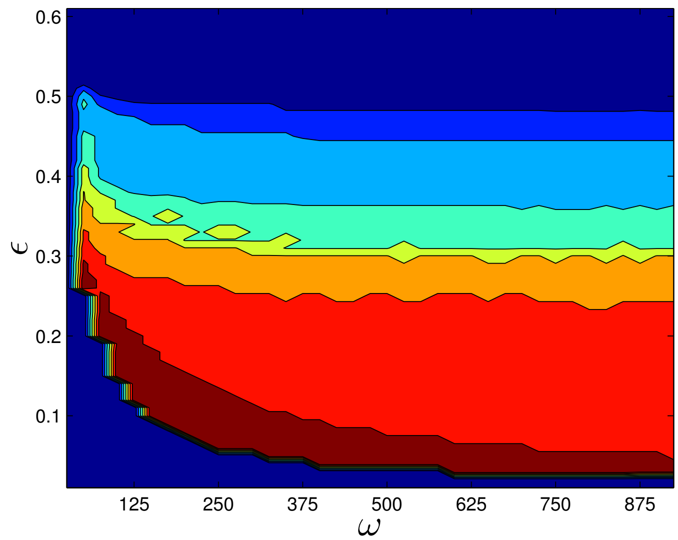


Figure 3: Contour plot of the width of the region of initial stability. Red regions correspond to wide stable region and blue correspond to narrow stable region.

3 More than one periodic frequency

Before adding noise, an initial test to see whether the inverted system is stable to more than one periodic drive. With $\epsilon_1 = \epsilon_2 = 0.1$ and $\omega_1 = 250$, ω_2 was varied. No value of ω_2 was found that caused the system to become unstable. Figure 4 shows a typical result of this simulation.

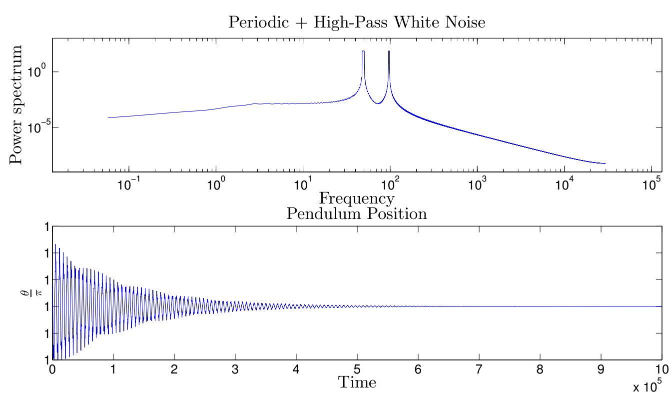


Figure 4: Results of a MatLab simulation with two periodic driving functions. Note that the inverted position is stable.

4 MatLab simulation – stochastic drive

Although there are many places in the literature that allude to stability through a noisy drive, there are not many places where it has in fact been shown. One such paper shows that in the case of sufficiently high-frequency noise the $\theta = \pi$ solution becomes stable. [6]

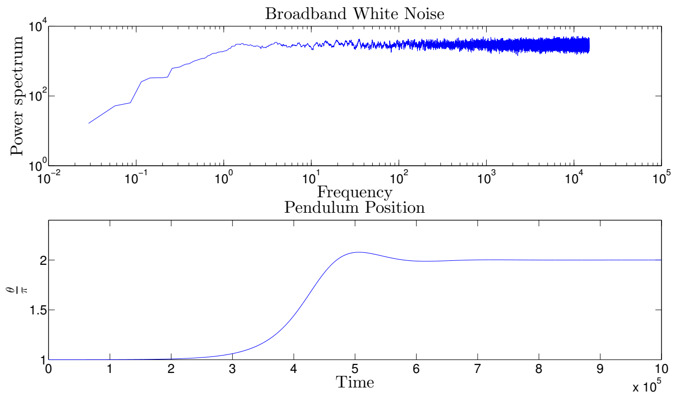


Figure 5: Results of a MatLab simulation with broadband white noise as the driving function. Note that the inverted position is not stable.

In order to investigate the stability through noise, the previous MatLab code was modified to incorporate uncorrelated broadband noise (see Appendix B for code). The simulations implied that broadband noise did not stabilize the inverted pendulum. Figure 5 shows a typical result of the broadband simulation.

MatLab simulation – high-pass filtered noisy drive

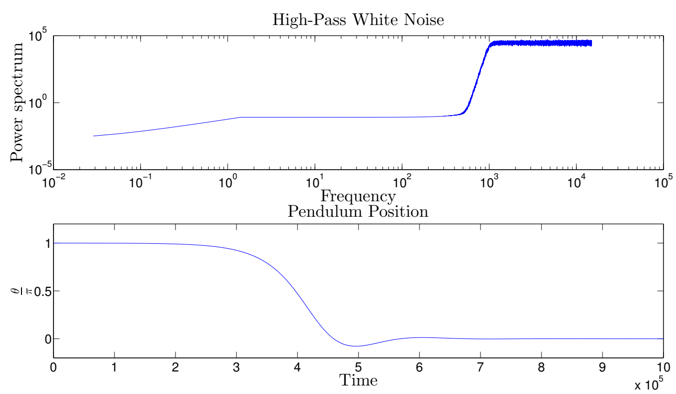
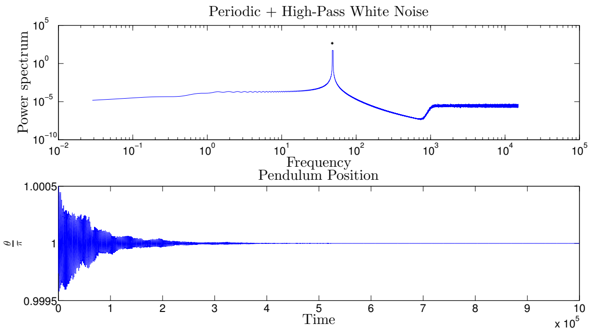


Figure 6: Results of a MatLab simulation with only high frequencies present in white noise as the driving function. Note that the inverted position is not stable.

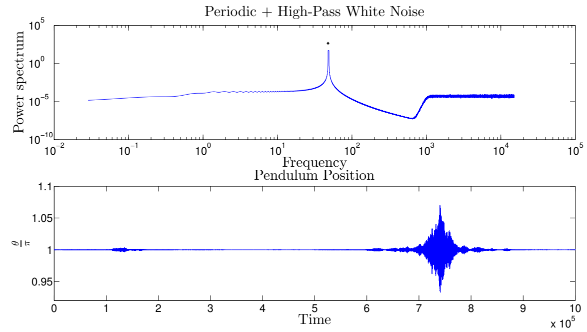
Landas theoretical work suggested that noise that was comprised of only high-frequency components is necessary for stability.[6] The modified MatLab code in Appendix B also was able to produce only high-frequency uncorelated noise by passing broadband uncorrelated white noise through a 10^{th} order Butterworth high-pass filter. The cutoff frequency was selected so that only frequencies above ω_{stable} where present. Figure 6 shows a typical result of a simulation. The MatLab simulation seems to imply that even pure high-frequency noise doesn't stabilize the inverted pendulum.

Noise and a periodic drive

To investigate the sensitivity to noise levels, κ was slowly increased. The values of ω and ϵ where set at the values which maximized the stability when no noise was present (*i.e.* $\omega = 250$ and $\epsilon = 0.1$). The stability of the system was observed to be extremely sensitive to the presence of noise. The maximal value of high-frequency noise that was allowed for stability to persist corresponded to $\frac{\epsilon}{\kappa} \approx 5.6 \times 10^7$.



(a) $\frac{\epsilon}{\kappa} = 10^9$



(b) $\frac{\epsilon}{\kappa} = 5.6 \times 10^7$

Figure 7: Results of a MatLab simulation with periodic drive and high frequency noise present. The astrix denotes the theoretical power in the periodic component. Note that the inverted position is stable.

5 Conclusions

There are not many references in the literature that actually treat the case of noisy inverted pendulums. Most of the discussions in the literature reference other works claiming "their results"; however, the papers they are citing often do not directly show stability exists. The only paper that I was able to find that directly found stability from a noisy drive was Landa 1997 [6]. Although Landa gave a derivation concluding that stability was possible through a noise drive, the result was not accompanied by a simulation or experiment.

In the simulations presented here, the nonlinear terms in the equation of motion were integrated to 2^{nd} order and the stochastic term was integrated to 1^{st} order. If the systems stability is robust to noisy drive than one would expect that a simplistic 1^{st} order treatment of the noise would suffice to recover stability. The inability of these simulations to produce stability indicates that the dynamical stability of an inverted pendulum is a subtle topic.

Future work

The results shown here are not comprehensive. The noise that was used to drive the pivot was broadband and high-pass white noise. An investigation of noise with a spectral band in a neighborhood around one frequency should be done before uncorrelated white noise be thrown out as a possible type of noise that would produce stability. Additionally, other forms of stochastic driving (*e.g.* phase-noise) should be investigated in a similar manor.

Appendix A

The MatLab simulation of a periodically shaking pivot with no noise:

```
function Driven_pendulum_PeriodicNoise_ver4(epsilon,w)
epsilon
w
dirname = sprintf('eps%06d_w%06d',round(epsilon*10000),round(w));
mkdir(dirname);
pause(10)
cd(dirname);
try
    matlabpool close force
end

alpha=linspace(0.001,pi/4,15);
matlabpool(6)
parfor i = 1:length(alpha) %half width of the initial spaceing
    std_alpha_w(i) = compute_final_std(alpha(i),w,epsilon);
end


save('stability.mat','epsilon','w','std_alpha_w','alpha');
cd ..

matlabpool close force

end
%%
function [measure] = compute_final_std(alpha,w,epsilon)
B=1;


%the freq im driving it at
dt=1/100/w; %makeing dt small enough to resolve the periodic drive

kappa=0; %amplitude of the noisey drive

n=1000000; %number of time steps
N=100; %number of realizations

theta_now = pi+linspace(-alpha,alpha,N);
theta_prev = theta_now;
theta_next = zeros(1,N);

omega_now = zeros(1,N);
omega__next = zeros(1,N);
noise = zeros(1,N);

for t=2:n
    noise = sqrt(3*kappa/dt)*(2*rand(1,N)-1);
    f = epsilon*sin(w*t*dt);
    f_DDnoise = 0;
    f_DD=-f.*w^2+f_DDnoise; %f double dot ?need to add noise_DD
    omega_next =  1/(1+B*dt/2)*((1-dt*B/2)*omega_now -dt*(1+f_DD).*(1.5*sin(theta_now)-0.5*sin
    theta_next = theta_now+dt*omega_next;

    theta_prev = theta_now;
```

```
    theta_now = theta_next;
    omega_now = omega_next;
end
measure=std(theta_now./(2*pi));
end
```

Appendix B

Sample MatLab simulation of a noisy pendulum where the noise has been put through a high-pass filter:

```
clear all
close all
clc
B=1;
w=12*25;
epsilon=0.1;
kappa=0.0000000001;
cut=1; % 1=send noise through High-Pass filter, 0=do not.
Fc=1000; %cut-off
epsilon/kappa
%the freq im driving it at
dt=1/100/w; %makeing dt small enough to resolve the periodic drive


n=1000000; %number of time steps
N=1; %number of realizations

theta_now = pi-0.001*(2*rand-1);
theta_prev = theta_now;
theta_next = zeros(1,N);

omega_now = zeros(1,N);
omega__next = zeros(1,N);
noise = zeros(n,N);
noise_raw = zeros(n,N); f=zeros(n,N);
x=zeros(1,n); x(1)=theta_prev; x(2)=theta_now;

Fs=1/dt;
[Bf,Af]=butter(10,2*Fc/Fs,'high'); %apply a high-pass filter
for t=1:n %generate the time dependent functions
    noise_raw(t,:) = sqrt(3*kappa/dt)*(2*rand(1,N)-1);
    f(t,:) = epsilon*sin(w*t*dt);
end
noise_raw(end+1)=sqrt(3*kappa/dt)*(2*rand(1,N)-1);% noise needs to be 2 longer
noise_raw(end+1)=sqrt(3*kappa/dt)*(2*rand(1,N)-1);% becouse i am takeing diff(diff())
if cut==0
    noise=noise_raw;
else
    noise=filter(Bf,Af,noise_raw);
end
DD_noise=diff(diff(noise))./(dt^2);%second derivative of noise?
noiseplot=noise(1:(end-2));

NFFT = 2^nextpow2(n); % Next power of 2 from length of y
PS=abs(fft(noiseplot+f,NFFT).^2)./n;
PS=PS(1:NFFT/2+1);
windowsize = 50;
bb=(1/windowsize)*ones(1,windowsize);
pspf=filter(bb,1,PS);
subplot(2,1,1)
loglog(Fs/2*linspace(0,1,NFFT/2+1),pspf)
hold on
loglog(w/2/pi,(w^2)*(epsilon^2)/(2),'*k')
xl=xlabel('Frequency');
```

```
yl=ylabel('Power spectrum');
tit=title('Periodic + High-Pass White Noise');
set(xl,'Interpreter','Latex','FontSize',32,'FontWeight','bold');
set(yl,'Interpreter','Latex','FontSize',32,'FontWeight','bold');
set(tit,'Interpreter','Latex','FontSize',32,'FontWeight','bold');

%%
for t=2:n
    f_DD=-f(t,:).*w^2+DD_noise(t,:); %f double dot ?need to add noise_DD
    omega_next =  1/(1+B*dt/2)*((1-dt*B/2)*omega_now -dt*(1+f_DD).*(1.5*sin(theta_now)-0.5*sir
    theta_next = theta_now+dt*omega_next;

    theta_prev = theta_now;
    theta_now = theta_next;
    omega_now = omega_next;
    x(t)=theta_now;
end
%%
subplot(2,1,2)
plot(x./(pi))
xl=xlabel('Time');
yl=ylabel('$\frac{\theta}{\pi}$');
set(xl,'Interpreter','Latex','FontSize',32,'FontWeight','bold');
set(yl,'Interpreter','Latex','FontSize',32,'FontWeight','bold');
tit=title('Pendulum Position');
set(tit,'Interpreter','Latex','FontSize',32,'FontWeight','bold');
```

References

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